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CONTINUOUS DEPENDENCE OF THE FINAL MOMENT OF EXISTENCE FOR THE SOLUTIONS OF DIFFERENTIAL EQUATIONS

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ABSTRACT

The main object of investigation is an initial problem of nonlinear non-autonomous system of differential equations. In the phase space of the system considered a final set of the existence of its solutions is situated. The moment, at which the solution of initial problem reaches the final set, is called a final moment. At this moment the solution terminates its existence. The sufficient conditions for the continuous dependence of the final moments of existence of the solutions to the effects associated with the initial conditions and final set are found.

Keywords: *differential equations, final (switching) set, final (switching) moment, continuous dependence*

INTRODUCTION

The impulsive equations are used mainly for describing and study the development of dynamic processes, subjected to the discrete external influences over time: [1-3], [7], [9-12]. These equations are divided into several classes, depending on the method of determining the impulsive moments. In one of these basic classes, the impulsive moments are determined using the switching sets, situated in the phase space. (see [4], [5] and [8]). The impulsive moments in these equations coincide with the moments in which the solution reaches the corresponding switching set.

The investigations in the paper are closely related to the qualitative theory of the impulsive differential equations, described above.

The object of research is the initial problem for nonlinear non-autonomous systems of ordinary differential equations:

$$\frac{dx}{dt} = f\left(t, x\right) \quad for \quad \varphi\left(x\left(t\right)\right) \neq 0, \tag{1}$$

$$x\left(t_0\right) = x_0, \tag{2}$$

where: the function $f : R^+ \times D \to R^n$, $f = \left(f^1, f^2, ..., f^n\right)$, the phase space D is non empty domain in R^n, the final (switching) function $\varphi : D \to R$, the initial point $\left(t_0, x_0\right) \in R^+ \times D$ and $\varphi\left(x_0\right) \neq 0$.

With the main problem, we discuss the so-called perturbed problem

$$\frac{dx}{dt} = f\left(t, x\right) \quad for \quad \varphi^*\left(x\left(t\right)\right) \neq 0, \tag{3}$$

$$x\left(t_0^*\right) = x_0^*, \tag{4}$$

where: the perturbed switching function $\varphi^* : D \to R$, the perturbed initial point $\left(t_0^*, x_0^*\right) \in R^+ \times D$ and $\varphi^*\left(x_0^*\right) \neq 0$.

The solutions of both problems above are denoted by $x\left(t; t_0, x_0, \varphi\right)$ and $x\left(t; t_0^*, x_0^*, \varphi^*\right)$, their trajectories are $\gamma\left(t_0, x_0, \varphi\right)$ and $\gamma\left(t_0^*, x_0^*, \varphi^*\right)$, their final (switching) sets are $\Phi = \left\{x = \left(x^1, ..., x^n\right) \in D; \varphi\left(x\right) = 0\right\}$ and

$\Phi^{*}=\{x\in D;\varphi^{*}(x)=0\}$, respectively. The moments t_1 and t_1^*, which satisfy the next equations:

$$\varphi\left(x\left(t_1;t_0,x_0,\varphi\right)\right)=0,$$

$$\varphi^{*}\left(x\left(t_1^{*};t_0^{*},x_0^{*},\varphi^{*}\right)\right)=0,$$

are called final (switching) moments for problems (1), (2) and (3), (4), respectively.

Definition 1. *We say that the switching moment of initial problem (1), (2) depends continuously on the initial condition and the switching function, if*

$$(\forall\varepsilon>0)\left(\exists\delta=\delta(\varepsilon),\ 0<\delta\le\varepsilon\right):$$
$$\left(\forall t_0^{*}\in R^{+},\ \left|t_0^{*}-t_0\right|<\delta\right)$$
$$\left(\forall x_0^{*}\in D,\ \left\|x_0^{*}-x_0\right\|<\delta\right)$$
$$\left(\forall\varphi^{*}\in C[D,R],\right.$$
$$\left.\left|\varphi^{*}(x)-\varphi(x)\right|<\delta\ \text{for}\ x\in D\right)$$
$$\Rightarrow\left|t_1^{*}-t_1\right|<\varepsilon.$$

We introduce the following conditions:

H1. *The function* $f\in C\left[R^{+}\times D,R^{n}\right]$.
H2. *The function* $\varphi\in C^{1}[D,R]$.
H3. *There exist a constant* $C_{Lip\varphi}>0$, *such that*

$$(\forall x',\ x''\in D)$$
$$\Rightarrow\left|\varphi(x')-\varphi(x'')\right|\le C_{Lip\varphi}\left\|x'-x''\right\|.$$

H4. *The following inequality is valid:*

$$\varphi(x).\left\langle grad\varphi(x),\ f(t,x)\right\rangle<0$$
for $(t,x)\in R^{+}\times(D\setminus\Phi)$.

H5. *There exist a constant* $C_{\langle grad\varphi,f\rangle}>0$, *such that*

$$\left(\forall(t,x)\in R^{+}\times D\right)$$
$$\Rightarrow\left|\left\langle grad\varphi(x),\ f(t,x)\right\rangle\right|\ge C_{\langle grad\varphi,f\rangle}.$$

H6. *For any point* $(t_0,x_0)\in R^{+}\times(D\setminus\Phi)$, *the solution* $x(t;t_0,x_0)$ *of initial problem*

$$\frac{dx}{dt}=f(t,x),\quad x(t_0)=x_0 \tag{5}$$

exists and is unique for $t\ge t_0$.

MAIN RESULTS

Theorem 1. [8] *Let the conditions H1, H2, H4, H5 and H6 be valid.*

Then the trajectory of problem (1), (2) meets the set Φ.

Theorem 2. *Let the conditions H1, H2, H4, H5 and H6 be valid.*

Then

$$(\exists\delta=const>0):$$
$$\left(\forall t_0^{*}\in R^{+},\ \left|t_0^{*}-t_0\right|<\delta\right)$$
$$\left(\forall x_0^{*}\in D,\ \left\|x_0^{*}-x_0\right\|<\delta\right)$$
$$\left(\forall\varphi^{*}\in C[D,R],\right.$$
$$\left.\left|\varphi^{*}(x)-\varphi(x)\right|<\delta\ \text{for}\ x\in D\right)$$
$$\Rightarrow\gamma\left(t_0^{*},x_0^{*},\varphi^{*}\right)\cap\Phi^{*}\neq\varnothing,$$

i.e. the trajectory of perturbed problem (3), (4) meets the perturbed switching set Φ^{*}.

Proof. According to Theorem 1, the trajectory $\gamma(t_0,x_0,\varphi)$ of problem (1), (2) meets the set Φ at the moment t_1. Under conditions H6 and H5 the next inequalities are valid:

$$\varphi(x_0)\neq0\ \text{and}\ \left\langle grad\varphi(x),\ f(t,x)\right\rangle\neq0$$
$$\text{for}\ (t,x)\in R^{+}\times D. \tag{6}$$

Further, we assume that $\varphi(x_0)<0$, whence it follows:

$$\varphi\left(x\left(t_0;t_0,x_0,\varphi\right)\right)<0,\ \varphi\left(x\left(t;t_0,x_0,\varphi\right)\right)<0$$
$$\text{for}\ t_0<t<t_1$$
$$\text{and}\ \varphi\left(x\left(t_1;t_0,x_0,\varphi\right)\right)=0.$$

The case $\varphi(x_0)>0$ is considered similarly.

First, we assume that $\phi(t)=\varphi\left(x\left(t;t_0,x_0\right)\right)\le0$ for each $t\ge t_0$, where

$x(t;t_0,x_0)$ is the solution of problem (5). Then t_1 is the point of maximum for the function ϕ. Hence

$$0=\frac{d}{dt}\phi(t_1)=\frac{d}{dt}\varphi(x(t_1;t_0,x_0))$$
$$=\frac{\partial}{\partial x^1}\varphi(x(t_1;t_0,x_0))\frac{d}{dt}x^1(t_1;t_0,x_0)$$
$$+\frac{\partial}{\partial x^2}\varphi(x(t_1;t_0,x_0))\frac{d}{dt}x^2(t_1;t_0,x_0)$$
$$+\dots\dots\dots\dots+$$
$$+\frac{\partial}{\partial x^n}\varphi(x(t_1;t_0,x_0))\frac{d}{dt}x^n(t_1;t_0,x_0)$$
$$=\frac{\partial}{\partial x^1}\varphi(x(t_1;t_0,x_0))f^1(t_1,x(t_1;t_0,x_0))$$
$$+\frac{\partial}{\partial x^2}\varphi(x(t_1;t_0,x_0))f^2(t_1,x(t_1;t_0,x_0))$$
$$+\dots\dots\dots\dots+$$
$$+\frac{\partial}{\partial x^n}\varphi(x(t_1;t_0,x_0))f^n(t_1,x(t_1;t_0,x_0))$$
$$=\left\langle grad\varphi(x(t_1,t_0,x_0)),\right.$$
$$\left. f(t_1,x(t_1;t_0,x_0))\right\rangle.$$

The last equality contradicts the second of inequalities (6). Therefore, there exists a point $\tau > t_1$ such that $\phi(\tau) =\varphi(x(\tau;t_0,x_0))>0$. From the conti-nuity of function ϕ for $t\ge t_0$, the inequality above and the assumption $\phi(t_0)=\varphi(x(t_0;t_0,x_0))=\varphi(x_0)<0$, it follows that there exists a positive constant δ_ϕ such that:

$$(\forall x\in B_{\delta_\phi}(x_0))\Rightarrow \varphi(x)<0$$

and

$$(\forall x\in B_{\delta_\phi}(x(\tau;t_0,x_0)))\Rightarrow \varphi(x)>0.$$

Let:

$$\Delta'=\inf\{|\varphi(x)|;\ x\in B_{\delta_\phi}(x_0)\}$$

and

$$\Delta''=\inf\{|\varphi(x)|;\ x\in B_{\delta_\phi}(x(\tau;t_0,x_0))\}.$$

According to the theorem of continuous dependence of the solutions of systems differential equations on the initial condition (see Theorem 7.1, § 7, Chapter I, [6] – for the brevity, we shall call theorem on continuous dependence), it follows:

$$(\exists\delta,\ 0<\delta<\delta_\phi):(\forall t_0^*\in R^+,\ |t_0^*-t_0|<\delta)$$
$$(\forall x_0^*\in D,\ \|x_0^*-x_0\|<\delta)$$
$$\Rightarrow(\|x(t;t_0^*,x_0^*)-x(t;t_0,x_0)\|<\delta_\phi$$
$$for\ t_0^{\max}=\max\{t_0^*,t_0\}\le t\le\tau) \tag{7}$$

Suppose that for the continuous function $\varphi^*:D\to R$, the inequality below is valid

$$|\varphi^*(x)-\varphi(x)|<\min\{\Delta',\Delta''\}\ for\ x\in D.$$

As $\|x_0^*-x_0\|<\delta<\delta_\phi$, we conclude that $x_0^*\in B_{\delta_\phi}(x_0)$. So, we have

$$\varphi^*(x_0^*)=\varphi^*(x_0^*)-\varphi(x_0^*)+\varphi(x_0^*)$$
$$\le|\varphi^*(x_0^*)-\varphi(x_0^*)|+\varphi(x_0^*) \tag{8}$$
$$=|\varphi^*(x_0^*)-\varphi(x_0^*)|-|\varphi(x_0^*)|$$
$$<\Delta'-\Delta'=0.$$

From (7) for $t=\tau$ it follows $x(\tau;t_0^*,x_0^*)\in B_{\delta_\phi}(x(\tau;t_0,x_0))$. Therefore

$$\varphi^*(x(\tau;t_0^*,x_0^*))=\varphi^*(x(\tau;t_0^*,x_0^*)) \tag{9}$$
$$-\varphi(x(\tau;t_0^*,x_0^*))+\varphi(x(\tau;t_0^*,x_0^*))$$
$$\ge|\varphi(x(\tau;t_0^*,x_0^*))|$$
$$-|\varphi^*(x(\tau;t_0^*,x_0^*))-\varphi(x(\tau;t_0^*,x_0^*))|$$
$$>\Delta''-\Delta''=0.$$

Using (8) and (9), we find that there exists a point $t_1^*,\ t_0^*<t_1^*<\tau$, such that $\varphi^*(x(t_1^*;t_0^*,x_0^*))=0$. The last equality means that the trajectory $\gamma(t_0^*,x_0^*)=\{x(t;t_0^*,x_0^*),\ t\ge t_0^*\}$ meets set Φ^* at moment t_1^*. As

$$\gamma(t_0^*,x_0^*,\varphi^*)=\gamma(t_0^*,x_0^*),\quad t_0^*\le t\le t_1^*,$$

we deduce that the trajectory of perturbed initial problem (3), (4) meets the perturbed switching set Φ^* at moment t_1^*.

The theorem is proved.

Theorem 3. *Let the conditions H1, H2, H4, H5 and H6 be valid.*

Then the next inequality is valid

$$t_1 - t_0 \le \frac{1}{C_{\langle grad\varphi, f\rangle}} \left|\varphi(x_0)\right|. \quad (10)$$

Proof. Under the assumption made (see condition H6) $x_0 \in D \setminus \Phi$ is fulfilled. Then, there exists a point τ, $t_0 < \tau < t_1$, such that:

$$\begin{aligned}
&\left|\varphi(x_0)\right| \\
&= \left|\varphi\left(x(t_1;t_0,x_0,\varphi)\right) - \varphi\left(x(t_0;t_0,x_0,\varphi)\right)\right| \\
&= \left|\frac{d}{dt}\varphi\left(x(\tau;t_0,x_0,\varphi)\right).(t_1 - t_0)\right| \\
&= \left|\left\langle grad\left(\varphi\left(x(\tau;t_0,x_0,\varphi)\right)\right),\right.\right. \\
&\left.\left. f\left(\tau, x(\tau;t_0,x_0,\varphi)\right)\right\rangle\right|.\left|t_1 - t_0\right| \\
&\ge C_{\langle grad\varphi, f\rangle}(t_1 - t_0),
\end{aligned}$$

whence it follows (10).

The theorem is proved.

Theorem 4. *Let the conditions H1-H6 be valid.*

Then the following inequality is true

$$t_1 - t_0 \le \frac{C_{Lip\varphi}}{C_{\langle grad\varphi, f\rangle}} \rho(x_0, \Phi). \quad (11)$$

Proof. Like the proof in the previous theorem, we assume that the initial point $x_0 \in D \setminus \Phi$. Under Theorem 1, the solution $x(t;t_0,x_0,\varphi)$ of problem (1), (2) cancels the function φ at moment t_1, i.e. the moment t_1 exists. Let ε be an arbitrary positive constant. Let the point $x_\varepsilon \in \Phi$, i.e. $\varphi(x_\varepsilon) = 0$ and $\rho(x_0, x_\varepsilon) = \left\|x_0 - x_\varepsilon\right\| \le \rho(x_0, \Phi) + \varepsilon$. We again denote $\phi(t) = \varphi\left(x(t;t_0,x_0)\right)$ for $t \ge t_0$. Recall that $x(t;t_0,x_0)$ is the solution of problem (5). It is also satisfied

$$x(t;t_0,x_0,\varphi) = x(t;t_0,x_0), \quad t_0 \le t \le t_1,$$

whence, we deduce that

$$\phi(t_1) = \varphi\left(x(t_1;t_0,x_0)\right)$$

$$= \varphi\left(x(t_1;t_0,x_0,\varphi)\right) = 0 \cdot$$

We assume (based on condition H4) that $\left\langle grad\varphi(x), f(t,x)\right\rangle > 0$, $(t,x) \in R^+ \times D$ *and* $\varphi(x_0) < 0$. (12)

Then $\phi(t_0) = \varphi(x_0) < 0$. Using condition H5 for $t > t_0$, we find that

$$\begin{aligned}
&\frac{d}{dt}\phi(t) \\
&= \left\langle grad\varphi\left(x(t;t_0,x_0)\right), f\left(t, x(t;t_0,x_0)\right)\right\rangle \\
&= \left|\left\langle grad\varphi\left(x(t;t_0,x_0)\right), f\left(t, x(t;t_0,x_0)\right)\right\rangle\right| \\
&\ge C_{\langle grad\varphi, f\rangle} \\
&= const > 0.
\end{aligned}$$

There is a point τ, $t_0 < \tau < t_1$, such that

$$\begin{aligned}
&\left|\phi(t_1) - \phi(t_0)\right| \\
&= \left|\frac{d}{dt}\phi(\tau)\right|(t_1 - t_0) \ge C_{\langle grad\varphi, f\rangle}(t_1 - t_0).
\end{aligned}$$

From the inequality above, we obtain successively

$$\begin{aligned}
t_1 - t_0 &\le \frac{1}{C_{\langle grad\varphi, f\rangle}}\left|\phi(t_1) - \phi(t_0)\right| \\
&= \frac{1}{C_{\langle grad\varphi, f\rangle}}\left|-\varphi\left(x(t_0;t_0,x_0)\right)\right| \\
&= \frac{1}{C_{\langle grad\varphi, f\rangle}}\left|\varphi(x_0)\right| \\
&= \frac{1}{C_{\langle grad\varphi, f\rangle}}\left|\varphi(x_0) - \varphi(x_\varepsilon)\right| \\
&\le \frac{C_{Lip\varphi}}{C_{\langle grad\varphi, f\rangle}}\left\|x_0 - x_\varepsilon\right\| \\
&\le \frac{C_{Lip\varphi}}{C_{\langle grad\varphi, f\rangle}}\left(\rho(x_0, \Phi) + \varepsilon\right) \cdot
\end{aligned}$$

Since ε is an arbitrary constant, it follows that (11) is satisfied.

The theorem is proved.

Theorem 5. *Let the conditions H1-H6 be valid. Then* $(\exists\delta = const > 0)$ *such that:*

$$\left(\forall t_0^* \in R^+,\ \left|t_0^* - t_0\right| < \delta\right) \tag{13}$$
$$\left(\forall x_0^* \in D,\ \left\|x_0^* - x_0\right\| < \delta\right)$$
$$\left(\forall \varphi^* \in C[D,R],\right.$$
$$\left.\left|\varphi^*(x) - \varphi(x)\right| < \delta \ \ for\ \ x \in D\right)$$

the following inequality is valid

$$t_1^* - t_0^* \le \frac{C_{Lip\varphi}}{C_{\langle grad\varphi, f\rangle}}\left(\rho\left(x_0^*, \Phi^*\right) + 2\delta\right) \tag{14}$$

Proof. According to Theorem 2, if the constant δ is sufficiently small and if the inequalities (13) are fulfilled, then the solution $x\left(t; t_0^*, x_0^*, \varphi^*\right)$ of problem (3), (4), cancels the function φ^* at moment t_1^*. As in the previous theorem, on the basis of condition H4, we assume that inequalities (12) are fulfilled.

We introduce the auxiliary notations:

$$\varphi^{-\delta}(x) = \varphi(x) - \delta, \quad x \in D,$$
$$\varphi^{+\delta}(x) = \varphi(x) + \delta, \quad x \in D,$$
$$\Phi^{-\delta} = \left\{x \in D,\ \varphi^{-\delta}(x) = 0\right\},$$
$$\Phi^{+\delta} = \left\{x \in D,\ \varphi^{+\delta}(x) = 0\right\}.$$

Taking into account the inequalities (13), we conclude that

$$\varphi^{-\delta}(x) = \varphi(x) - \delta < \varphi^*(x)$$
$$< \varphi(x) + \delta = \varphi^{+\delta}(x), \quad x \in D.$$

From the first inequality of (12), we establish that

$$(\exists\delta > 0) : \left(\forall x,\ \left\|x - x_0\right\| \le \delta\right)$$
$$\Rightarrow x \in D \ \ and \ \ \varphi(x) < 0.$$

Having in mind (13), we arrive at the conclusion

$$\varphi\left(x_0^*\right) = \varphi\left(x\left(t_0^*; t_0^*, x_0^*\right)\right) < 0 \tag{15}$$

Consider the function $\phi^{-\delta} : \left[t_0^*, \infty\right) \to R$, defined by the equality

$$\phi^{-\delta}(t) = \varphi^{-\delta}\left(x\left(t; t_0^*, x_0^*\right)\right)$$
$$= \varphi\left(x\left(t; t_0^*, x_0^*\right)\right) - \delta$$

Under inequality (15), it follows that

$$\phi^{-\delta}\left(t_0^*\right) = \varphi^{-\delta}\left(x\left(t_0^*; t_0^*, x_0^*\right)\right)$$
$$= \varphi\left(x\left(t_0^*; t_0^*, x_0^*\right)\right) - \delta$$
$$= \varphi\left(x_0^*\right) - \delta < \varphi\left(x_0^*\right) < 0.$$

As above, we find that:

$$\frac{d}{dt}\phi^{-\delta}(t) = \frac{d}{dt}\phi(t) \ge C_{\langle grad\varphi, f\rangle} > 0,\ t > t_0^*.$$

Then, there exists a moment $t_1^{-\delta}$, $t_1^{-\delta} > t_0^*$, such that $\phi^{-\delta}\left(t_1^{-\delta}\right) = 0$, i.e. the solution $x\left(t; t_0^*, x_0^*\right)$ cancels function $\varphi^{-\delta}$ at the moment $t_1^{-\delta}$.

We obtain the next estimate using the previous theorem 4

$$t_1^{-\delta} - t_0^* \le \frac{C_{Lip\varphi}}{C_{\langle grad\varphi, f\rangle}}\rho\left(x_0^*, \Phi^{-\delta}\right) \tag{16}$$
$$\le \frac{C_{Lip\varphi}}{C_{\langle grad\varphi, f\rangle}}\left(\rho\left(x_0^*, \Phi^*\right) + \rho\left(\Phi^*, \Phi^{-\delta}\right)\right)$$
$$\le \frac{C_{Lip\varphi}}{C_{\langle grad\varphi, f\rangle}}\left(\rho\left(x_0^*, \Phi^*\right) + \rho\left(\Phi^{-\delta}, \Phi^{+\delta}\right)\right)$$
$$\le \frac{C_{Lip\varphi}}{C_{\langle grad\varphi, f\rangle}}\left(\rho\left(x_0^*, \Phi^*\right) + 2\delta\right).$$

On the one hand, we have

$$\varphi^*\left(x\left(t_0^*; t_0^*, x_0^*\right)\right) < 0.$$

On the other hand, it is satisfied

$$\varphi^*\left(x\left(t_1^{-\delta}; t_0^*, x_0^*\right)\right) - \delta = 0$$
$$\Rightarrow \varphi^*\left(x\left(t_1^{-\delta}; t_0^*, x_0^*\right)\right) > 0.$$

The above two inequalities imply inequality $t_0^* < t_1^* < t_1^{-\delta}$. Therefore, $t_1^* - t_0^* < t_1^{-\delta} - t_0^*$. Whence, using (16) we reach (14).

The theorem is proved.

Theorem 6. *Let the conditions H1-H6 be valid. Then*

$$(\forall \omega = const > 0)\left(\exists \delta = \delta(\omega) > 0\right):$$
$$\left(\forall t_0^* \in R^+,\ \left|t_0^* - t_0\right| < \delta\right)$$
$$\left(\forall x_0^* \in D,\ \left\|x_0^* - x_0\right\| < \delta\right)$$
$$\Rightarrow \left|t_1^{*\varphi} - t_1\right| \le \frac{C_{Lip\varphi}}{C_{\langle grad\varphi, f\rangle}}\omega,$$

where $t_1^{*\varphi}$ is the moment, at which solution $x\left(t;t_0^*,x_0^*\right)$ cancels switching function φ, and t_1 is the moment at which solution $x\left(t;t_0,x_0\right)$ (or which is also, solution $x\left(t;t_0,x_0,\varphi\right)$) cancels the same switching function.

Proof. As a result of Theorem 1, we obtain that every one of the solutions $x\left(t;t_0^*,x_0^*\right)$ and $x\left(t;t_0,x_0\right)$ cancels switching function φ, i.e. the moments $t_1^{*\varphi}$ and t_1 exist.

For the convenience, we suppose that $t_1 \le t_1^{*\varphi}$ is satisfied. Let ω be an arbitrary positive constant. From the theorem of continuous dependence, it follows that there exists constant $\delta > 0$ such that, if the requirements of the proving theorem are satisfied, then it follows

$$\rho\left(x\left(t_1;t_0^*,x_0^*\right), x\left(t_1;t_0,x_0\right)\right)$$
$$= \left\|x\left(t_1;t_0^*,x_0^*\right) - x\left(t_1;t_0,x_0\right)\right\| \le \omega.$$

We have

$$\rho\left(x\left(t_1;t_0^*,x_0^*\right), \Phi\right)$$
$$\le \rho\left(x\left(t_1;t_0^*,x_0^*\right), x\left(t_1;t_0,x_0\right)\right) \le \omega.$$

Applying Theorem 4, we reach the estimate

$$\left|t_1^{*\varphi} - t_1\right| = t_1^{*\varphi} - t_1$$
$$\le \frac{C_{Lip\varphi}}{C_{\langle grad\varphi, f\rangle}} \rho\left(x\left(t_1;t_0^*,x_0^*\right), \Phi\right)$$
$$\le \frac{C_{Lip\varphi}}{C_{\langle grad\varphi, f\rangle}}\omega.$$

The theorem is proved.

Theorem 7. *Let the conditions H1-H6 be valid. Then*

$$(\forall \omega = const > 0)$$
$$\left(\exists \delta = \delta(\omega) > 0\right)$$
$$\left(\exists \Theta = \Theta(x_0) > 0\right):$$
$$\left(\forall t_0^* \in R^+,\ \left|t_0^* - t_0\right| < \delta\right)$$
$$\left(\forall x_0^* \in D,\ \left\|x_0^* - x_0\right\| < \delta\right)$$
$$\left(\forall \theta,\ 0 \le \theta \le \Theta\right) \Rightarrow \left|t_1^{*+\theta} - t_1^{+\theta}\right| \le \frac{C_{Lip\varphi}}{C_{\langle grad\varphi, f\rangle}}\omega,$$

where $t_1^{*+\theta}$ is the moment at which the solution $x\left(t;t_0^*,x_0^*\right)$ cancels function $\varphi^{+\theta}(x) = \varphi(x) + \theta$, defined for $x \in D$, and $t_1^{+\theta}$ is the moment, at which solution $x\left(t;t_0,x_0\right)$ cancels the same function.

Proof. Let us again, for the convenience to assume that the inequality $\varphi(x_0) < 0$ is valid. Then there exists a positive constant $\Theta = \Theta(x_0)$, such that

$$\left(\forall \theta,\ 0 \le \theta \le \Theta\right)$$
$$\Rightarrow \varphi(x_0) + \theta = \varphi^{+\theta}(x_0) < 0.$$

From Theorem 1, we obtain that any of the solutions $x\left(t;t_0^*,x_0^*\right)$ and $x\left(t;t_0,x_0\right)$ cancels the function $\varphi^{+\theta}$ at the moments $t_1^{*+\theta}$ and $t_1^{+\theta}$, respectively. Further, the reasoning are as in the previous theorem. The inequality $t_1^{+\theta} \le t_1^{*+\theta}$ is valid and ω is an arbitrary positive constant. By the theorem of continuous dependence, it follows that there exists a positive constant δ, such that if the restrictions of the initial points $\left(t_0^*,x_0^*\right)$ and $\left(t_0,x_0\right)$ are satisfied, then

$$\rho\left(x\left(t_1^{+\theta};t_0^*,x_0^*\right), x\left(t_1^{+\theta};t_0,x_0\right)\right)$$
$$= \left\|x\left(t_1^{+\theta};t_0^*,x_0^*\right) - x\left(t_1^{+\theta};t_0,x_0\right)\right\| \le \omega.$$

Then, we have

$$\rho\left(x\left(t_1^{+\theta};t_0^*,x_0^*\right), \Phi\right)$$
$$\le \rho\left(x\left(t_1^{+\theta};t_0^*,x_0^*\right), x\left(t_1^{+\theta};t_0,x_0\right)\right) \le \omega.$$

We apply Theorem 4 and get the estimate in Theorem 7.

Theorem 8. *Let the conditions H1-H6 be valid.*

Then the switching moment of initial problem (1), (2) depends continuously on the initial condition and the switching function.

Proof. Let δ be an arbitrary positive constant. We consider the functions $\varphi, \varphi^*, \varphi^{-\delta}, \varphi^{+\delta} : D \to \mathbb{R}$, where:

- φ is a switching function;

- $\varphi^* \in C[D,R], \left|\varphi^*(x) - \varphi(x)\right| < \delta,$

  $x \in D$;

- $\varphi^{-\delta}(x) = \varphi(x) - \delta,\ x \in D$;

- $\varphi^{+\delta}(x) = \varphi(x) + \delta,\ x \in D$.

For any $x \in D$, we have:

$$\varphi^{-\delta}(x) \le \varphi(x) \le \varphi^{+\delta}(x),$$

$$\varphi^{-\delta}(x) \le \varphi^*(x) \le \varphi^{+\delta}(x).$$

As a result of Theorem 1, we obtain that if constant δ is sufficiently small and in particular, if the values:

$$\varphi^{+\delta}\left(x_0^*\right),\ \varphi\left(x_0^*\right),\ \varphi^{-\delta}\left(x_0^*\right),$$
$$\varphi^{+\delta}\left(x_0\right),\ \varphi\left(x_0\right) \text{ and } \varphi^{-\delta}\left(x_0\right)$$

have the same signs, then the solution $x\left(t;t_0^*,x_0^*\right)$ cancels functions $\varphi^{+\delta}$, φ^* and $\varphi^{-\delta}$, successively and solution $x\left(t;t_0,x_0\right)$, cancels one after another functions $\varphi^{+\delta}$, φ and $\varphi^{-\delta}$. Assume that, the values above are negative. Let the following equalities be satisfied:

$$\varphi^{+\delta}\left(x\left(t_1^{*+\delta};t_0^*,x_0^*\right)\right) = 0,$$

$$\varphi^{*}\left(x\left(t_1^{*};t_0^*,x_0^*\right)\right) = 0,$$

$$\varphi^{-\delta}\left(x\left(t_1^{*-\delta};t_0^*,x_0^*\right)\right) = 0,$$

$$\varphi^{+\delta}\left(x\left(t_1^{+\delta};t_0,x_0\right)\right) = 0,$$

$$\varphi\left(x\left(t_1;t_0,x_0\right)\right) = 0,$$

$$\varphi^{-\delta}\left(x\left(t_1^{-\delta};t_0,x_0\right)\right) = 0.$$

The moments, in which the functions referred above are canceled, satisfy the inequalities:

$$t_1^{*+\delta} \le t_1^* \le t_1^{*-\delta};\quad t_1^{+\delta} \le t_1 \le t_1^{-\delta} \tag{17}$$

Let ω be an arbitrary positive constant. We apply Theorem 7 for the solutions $x\left(t;t_0^*,x_0^*\right)$, $x\left(t;t_0,x_0\right)$ and function $\varphi^{-\delta}$ and obtain that

$$\left(\exists\delta = \delta(\omega) > 0\right):\ \left(\forall t_0^* \in R^+,\ \left|t_0^* - t_0\right| < \delta\right)$$
$$\left(\forall x_0^* \in D,\ \left\|x_0^* - x_0\right\| < \delta\right)$$
$$\left(\forall\theta,\ 0 \le \theta \le 2\delta = \Theta\right)$$
$$\Rightarrow \left|t_1^{*-\delta+\theta} - t_1^{-\delta+\theta}\right| \le \frac{C_{Lip\varphi}}{C_{\langle grad\varphi, f\rangle}}\omega \tag{18}$$

where
$t_1^{*-\delta+\theta}$ is the moment in which solution
$x\left(t;t_0^*,x_0^*\right)$ cancels function
$\varphi^{-\delta+\theta}(x) = \varphi(x) - \delta + \theta,\ x \in D,$
$t_1^{-\delta+\theta}$ is the moment in which solution
$x\left(t;t_0,x_0\right)$ cancels the same function. From (18), for $\theta = 0$ and $\theta = 2\delta$, we find out the next estimates, respectively:

$$\left|t_1^{*-\delta} - t_1^{-\delta}\right| \le \frac{C_{Lip\varphi}}{C_{\langle grad\varphi, f\rangle}}\omega, \tag{19}$$

$$\left|t_1^{*+\delta} - t_1^{+\delta}\right| \le \frac{C_{Lip\varphi}}{C_{\langle grad\varphi, f\rangle}}\omega.$$

By Theorem 3, more precisely from inequality (10) (after replacing in the theorem: φ with $\varphi^{-\delta}$; t_0 with $t_1^{+\delta}$; t_1 with $t_1^{-\delta}$ and x_0 with $x\left(t_1^{+\delta};t_0,x_0\right)$), we have

$$t_1^{-\delta} - t_1^{+\delta}$$

$$\leq \frac{1}{C_{\langle grad\varphi, f\rangle}} \left| \varphi^{-\delta}\left(x\left(t_1^{+\delta}; t_0, x_0\right)\right)\right|$$

$$= \frac{1}{C_{\langle grad\varphi, f\rangle}} \left| \varphi^{+\delta}\left(x\left(t_1^{+\delta}; t_0, x_0\right)\right)\right.$$

$$\left. -\varphi^{-\delta}\left(x\left(t_1^{+\delta}; t_0, x_0\right)\right)\right|$$

$$= \frac{1}{C_{\langle grad\varphi, f\rangle}} 2\delta . \qquad (20)$$

By the inequalities (17), (19) and (20), we obtain successively

$$\left|t_1^* - t_1\right| \leq \max\left\{t_1^{*-\delta}, t_1^{-\delta}\right\} - \min\left\{t_1^{*+\delta}, t_1^{+\delta}\right\}$$

$$\leq t_1^{-\delta} - t_1^{+\delta} + \left|t_1^{*-\delta} - t_1^{-\delta}\right| + \left|t_1^{*+\delta} - t_1^{+\delta}\right|$$

$$\leq \frac{1}{C_{\langle grad\varphi, f\rangle}} 2\delta + 2\frac{C_{Lip\varphi}}{C_{\langle grad\varphi, f\rangle}}\omega .$$

The theorem is proved.

REFERENCES

1. J. Angelova, A. Dishliev, S. Nenov, I–optimal curve for impulsive Lotka–Volterra predator–prey model, Computers & Mathematics with Applications, **43**, 10-11, 2002, 1203-1218.
2. J. Angelova, A. Dishliev, Optimi-zation problems in population dynamics, Applicable Analysis, **69**, 3 & 4, 1998, 207-221.
3. D. Bainov, A. Dishliev, I. Stamova, Lipschitz quasistability of impulsive differential–difference equations with variable inpulsive perturbations, J. of Computational and Applied Mathematics, **70**, 2, 1996, 267-277.
4. R. Chukleva, A. Dishliev, K. Dishlieva, Continuous dependence of the solutions of the differential equations with variable structure and impulses in respect of switching functions International J. of Applied Science and Technology, **1**, 5, 2011, 46-59.
5. R. Chukleva, Modeling using differential equations with variable structure and impulses, International Journal of Pure and Applied Mathematics, **72**, 3, 2011, 343-364.
6. E. Coddington, N. Levinson, Theory of ordinary differential equations, McGraw-Hill Book Company, New York, Toronto, London, 1955.
7. A. Dishliev, D. Bainov, Continuous dependence of the solution of a system of differential equations with impulses on the impulse hypersurfaces, J. of Math. Analysis and Applications, **135**, 2, 1988, 369-382.
8. K. Dishlieva, A. Dishliev, S. Petkova, Death of the solutions of systems differential equations with variable structure and impulses, International Journal of Differential Equations and Applications, **11**, 3, 2012, 169-181.
9. S. Nenov, Impulsive controllability and optimizations problems in population dynamics, Nonlinear Analysis, **36**, 7, 1999, 881-890.
10. L. Nie, Z. Teng, L. Hu, J. Peng, The dynamics of a Lotka-Volterra predator-prey model with state dependent impulsive harvest for predator, BioSystems, **98**, 2, 2009, 67-72.
11. Z. Shuai, L. Bai, K. Wang, Optimization problems for general simple population with n-impulsive harvest, J. of Mathematical Analysis and Applications, **329**, 1, 2007, 634-646.
12. H. Wang, E. Feng, Z. Xiu, Optimality control of the nonlinear impulsive system in fed-batch fermentation, Nonlinear Analysis: Theory, Methods & Applications, **68,** 1, 2008, 12-23.